Competition between collective and individual dynamics

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Linking microscopic and macroscopic behavior is at the heart of many natural and social sciences. This apparent similarity conceals essential differences across disciplines: Although physical particles are assumed to optimize the global energy, economic agents maximize their own utility. Here, we solve exactly a Schelling-like segregation model, which interpolates continuously between cooperative and individual dynamics. We show that increasing the degree of cooperativity induces a qualitative transition from a segregated phase of low utility toward a mixed phase of high utility. By introducing a simple function that links the individual and global levels, we pave the way to a rigorous approach of a wide class of systems, where dynamics are governed by individual strategies.

socioeconomy | statistical physics | segregation | phase transition | coordination

The intricate relations between the individual and collective levels are at the heart of many natural and social sciences. Different disciplines wonder how atoms combine to form solids (1, 2), neurons give rise to consciousness (3, 4), or individuals shape societies (5, 6). However, scientific fields assume distinct points of view for defining the "normal", or "equilibrium", aggregated state. Physics looks at the collective level, selecting the configurations that minimize the global free energy (2). In contrast, economic agents behave in a selfish way, and equilibrium is attained when no agent can increase its own satisfaction (7). Although similar at first sight, the two approaches lead to radically different outcomes.

In this paper, we illustrate the differences between collective and individual dynamics on an exactly solvable model similar to Schelling's segregation model (8). The model considers individual agents that prefer a mixed environment, with dynamics that lead to segregated or mixed patterns at the global level. A "tax" parameter monitors continuously the agents' degree of altruism or cooperativity, i.e., their consideration of the global welfare. At high degrees of cooperativity, the system is in a mixed phase of maximal utility. As the altruism parameter is decreased, a phase transition occurs, leading to segregation. In this phase, the agents' utilities remain low, in spite of continuous efforts to maximize their satisfaction. This paradoxical result of Schelling's segregation model (8) has generated an abundant literature. Many papers have simulated how the global state depends on specific individual utility functions, as reviewed by ref. 9. There have been attempts at solving Schelling's model analytically, in order to provide more general results (10, 11). However, these approaches are limited to specific utility functions. More recently, physicists have tried to use a statistical physics approach to understand the segregation transition (12–14). The idea seems promising because statistical physics has successfully bridged the micro–macro gap for physical systems governed by collective dynamics. However, progress was slowed by lack of an appropriate framework allowing for individual dynamics (12). In this paper, we introduce a rigorous generalization of the physicist's free energy, which includes individual dynamics. By introducing a "link" state function that is maximized in the stationary state, we pave the way to analytical treatments of a much wider class of systems where dynamics are governed by individual strategies. Applied to the above Schelling-like segregation model, this approach offers a quantitative solution for very general utility functions.

Model

Our model represents in a schematic way the dynamics of residential moves in a city. For simplicity, we include one type of agent, but our results can readily be generalized to deal with agents of two "colors," as in the original Schelling model (8) (see *Discussion* and *SI Appendix*). The city is divided into Q blocks ($Q \gg 1$), each block containing H cells or flats (Fig. 1). We assume that each cell can contain at most one agent, so that the number n_q of agents in a given block q ($q = 1, \ldots, Q$) satisfies $n_q \leq H$, and we introduce the density of agents $\rho_q = n_q/H$. Each agent has the same utility function $u(\rho_q)$, which describes the degree of satisfaction concerning the density of the block in which he is living. The collective utility is defined as the total utility of all the agents in the city: $U(x) = H \sum_q \rho_q u(\rho_q)$, where $x \equiv \{\rho_q\}$ corresponds to the coarse-grained configuration of the city, i.e., the knowledge of the density of each block. For a given x, there is a large number of ways to arrange the agents in the different cells. This number of arrangements is quantified by its logarithm $S(x)$, called the entropy of the configuration x.

The dynamical rule allowing the agents to move from one block to another is the following. At each time step, one picks up at random an agent and a vacant cell. Then the agent moves in that empty cell with probability

$$P_{xy} = \frac{1}{1 + e^{-\mathcal{G}/T}}, \qquad [1]$$

where x and y are respectively the configurations before and after the move, and $\mathcal{G}$ is the gain associated to the proposed move. The positive parameter T is a "temperature" that introduces in a standard way (15) some noise on the decision process. It can be interpreted as the effect of features that are not explicitly included in the utility function but still affect the moving decision (urban facilities, friends, etc.). We write the gain $\mathcal{G}$ as

$$\mathcal{G} = \Delta u + \alpha(\Delta U - \Delta u), \qquad [2]$$

where Δu is the variation of the agent's own utility upon moving and ΔU is the variation of the total utility of all agents. The parameter $0 \leq \alpha \leq 1$ weights the contribution of the other agents' utility variation in the calculation of the gain $\mathcal{G}$, and it can thus be interpreted as a degree of cooperativity (or altruism). For $\alpha = 0$, the probability to move only depends on the selfish interest of the chosen agent, which corresponds to the spirit of economic models such as Schelling's. When $\alpha = 1$, the decision to move only

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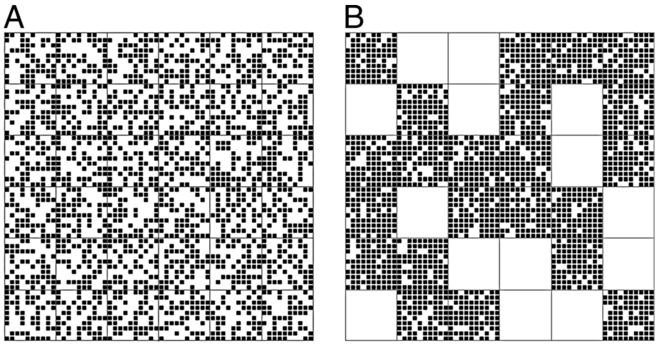

Fig. 1. Configurations of a city composed of $Q = 36$ blocks containing each $H = 100$ cells, with $\rho_0 = 1/2$. (*A*) Mixed state. Stationary state of the city for $m = 0.5$, $\alpha = 1$, and $T \to 0$. Agents are distributed homogeneously between the blocks, each of them having a density of 0.5. (*B*) Segregated configuration. Stationary state of the city for $m = 0.5$, $\alpha = 0$, and $T \to 0$. Agents are gathered on 22 blocks of mean density 0.82, the other blocks being empty. In the original Schelling model (8), each agent has a distinct neighborhood, defined by its eight nearest neighbors. Here, we only keep the essential ingredient of blocks of distinct densities. Our model shows the same qualitative behavior as Schelling's but can be solved exactly, thanks to the partial reduction of the agents' heterogeneity.

depends on the collective utility change, as in physics models. An economical interpretation could be that individual moves are controlled by a central government, via a tax that internalizes all the externalities (more on this in *Discussion*). Varying α in a continuous way, one can interpolate between the two limiting behaviors of individual and collective dynamics.

Results

We wish to find the stationary probability distribution $\Pi(x)$ of the microscopic configurations x. If the gain $\mathcal{G}$ can be written as $\mathcal{G} = \Delta V \equiv V(y) - V(x)$, where $V(x)$ is a function of the configuration x, then the dynamics satisfy the detailed balance (16), and the distribution $\Pi(x)$ is given by

$$\Pi(x) = \frac{1}{Z} e^{F(x)/T}, \qquad [3]$$

with $F(x) = V(x) + TS(x)$ and Z a normalization constant. The entropy has for large H the standard expression $S(x) = H \sum_q s(\rho_q)$, with

$$s(\rho) = -\rho \ln \rho - (1-\rho) \ln(1-\rho). \qquad [4]$$

We now need to find the function $V(x)$, if it exists. Given the form in Eq. **2** of $\mathcal{G}$, finding such a function $V(x)$ amounts to finding a "link" function $L(x)$, connecting the individual and collective levels, such that $\Delta u = \Delta L$. The function V would thus be given by $V(x) = (1-\alpha)L(x) + \alpha U(x)$. By analogy to the entropy, we assume that $L(x)$ can be written as a sum over the blocks, namely $L(x) = H \sum_q \ell(\rho_q)$. Considering a move from a block at density ρ_1 to a block at density ρ_2, ΔL reduces in the large H limit to $\ell'(\rho_2) - \ell'(\rho_1)$, where ℓ' is the derivative of ℓ. The condition $\Delta u = \Delta L$ then leads to the identification $\ell'(\rho) = u(\rho)$, from which the expression of $\ell(\rho)$ follows:

$$\ell(\rho) = \int_0^\rho u(\rho')d\rho'. \qquad [5]$$

As a result, the function $F(x)$ can be expressed in the large H limit as $F(x) = H \sum_q f(\rho_q)$, with a block potential $f(\rho)$ given by

$$f(\rho) = -T\rho \ln \rho - T(1-\rho)\ln(1-\rho) + \alpha\rho u(\rho) + (1-\alpha)\int_0^\rho u(\rho')d\rho'. \qquad [6]$$

The probability $\Pi(x)$ is dominated by the configurations $x = \{\rho_q\}$ that maximize the sum $\sum_q f(\rho_q)$ under the constraint of a fixed $\rho_0 = Q^{-1} \sum_{q=1}^{Q} \rho_q$. To perform this maximization procedure, we follow standard physics methods used in the study of phase transitions [like liquid–vapor coexistence (18)], which can be summarized as follows. If $f(\rho)$ coincides with its concave hull at a given density ρ_0, then the state of the city is homogeneous, and all blocks have a density ρ_0. Otherwise, a phase separation occurs: Some blocks have a density $\rho_1^* < \rho_0$, whereas the others have a density $\rho_2^* > \rho_0$ (see *SI Appendix*).

Interestingly, the potential $F = (1-\alpha)L + \alpha U + TS$ appears as a generalization of the notion of free energy introduced in physical systems. Mapping the global utility U onto the opposite of the energy of a physical system, it turns out that for $\alpha = 1$, the maximization of the function $U + TS$ is equivalent to the minimization of the free energy $E - TS$. For $\alpha < 1$, the potential F takes into account individual moves through the link function L. Furthermore, the potential F can be calculated for arbitrary utility functions, allowing one to predict analytically the global town state. Such an achievement has thus far eluded individualistic, Schelling-type models, which had to be studied through numerical simulations (9).

To explicitly obtain the equilibrium configurations, one needs to know the specific form of the utility function. To illustrate the dramatic influence of the cooperativity parameter α, we use the asymmetrically peaked utility function (18), which indicates that agents prefer mixed blocks (Fig. 2). The overall town density is fixed at $\rho_0 = 1/2$ to avoid the trivial utility frustration resulting from the impossibility to attain the optimal equilibrium ($\rho_q = 1/2$ for all blocks). We also consider for simplicity the limit $T \to 0$ in order to avoid entropy effects. The qualitative behavior of the system is unchanged for $\rho_0 \neq 1/2$ or for low values of the temperature, as shown in the *SI Appendix*.

In the collective case ($\alpha = 1$), the optimal state corresponds to the configuration that maximizes the global utility, which can be immediately guessed from Fig. 2, namely $\rho_q = 1/2$ for all q (Fig. 1*A*). On the contrary, in the selfish case ($\alpha = 0$, Fig. 1*B*), maximization of the potential $F(x)$ shows that the town settles in a segregated configuration where a fraction of the blocks are empty and the others have a density $\rho_s > 1/2$. Surprisingly, the city settles in this state of low utility in spite of agents' continuous efforts to maximize their own satisfaction. To understand this frustrated configuration, note that the collective equilibrium ($\rho_q = 1/2$ for all q) is now an unstable Nash equilibrium at $T > 0$. The instability can be understood by noting that at $T > 0$ there

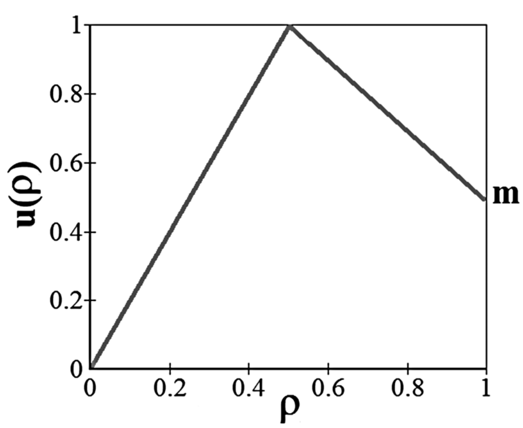

Fig. 2. Asymmetrically peaked individual utility as a function of block density. The utility is defined as $u(\rho) = 2\rho$ if $\rho \le 1/2$ and $u(\rho) = m + 2(1-m)(1-\rho)$ if $\rho > 1/2$, where $0 < m < 1$ is the asymmetry parameter. Agents strictly prefer half-filled neighborhoods ($\rho = 1/2$). The agents also prefer overcrowded ($\rho = 1$) neighborhoods to empty ones ($\rho = 0$).

Table 1. Characteristics of the segregated equilibrium

Quantity	$m < 2/3$	$2/3 \le m \le 1$
ρ_s	$\frac{1}{2}\sqrt{(2-m)/(1-m)}$	1
U^*	$\frac{1}{1+\sqrt{(1-m)/(2-m)}}$	m
L^*	$\frac{1}{1+\sqrt{(1-m)/(2-m)}}$	$1/2 + m/4$

The table displays the density ρ_s in the nonempty blocks, the normalized collective utility U^* and the normalized link L^* of the stationary configurations obtained for $\alpha = 0$. It is straightforward to check that $U^*(m) \le 1$ and $L^*(m) \ge 1/2$ for $m \le 1$.

is a positive probability that an agent accepts a slight decrease of its utility and leaves a block with density $\rho_q = 1/2$. The agents remaining in its former block now have a lower utility and are more likely to leave to go to another $\rho_q = 1/2$ block. This departure creates an avalanche that empties the block as each move away further decreases the utility of the remaining agents. This avalanche stops when the stable (Nash) equilibrium, given by the maximum of the potential, is reached. This state corresponds to a spatially inhomogeneous repartition of agents in the city. To understand the transition between mixed and segregated configurations, it is instructive to calculate the values of both the overall utility and the potential for different values of m (at $\alpha = 0$). For homogeneous towns, for all m, the normalized collective utility is given by $U^* = U/(\rho_0 HQ) = u(\rho_0 = 1/2) = 1$ and the normalized link function equals $L^* = L/(\rho_0 HQ) = \ell(\rho_0)/\rho_0 = 1/2$, where ℓ is given in Eq. **5**. The values of L^* and U^* displayed in Table 1 show that the utility of the segregated equilibrium is lower but that its potential is higher, explaining its stability. Note that the gap between the link function values of the homogeneous and segregated configurations increases with m.

This increase helps one understand why the greater the m, the greater the value of tax parameter necessary to reach the homogeneous configuration. Indeed, the segregated states are separated from mixed states by a phase transition at the critical value $\alpha_c = 1/(3-2m)$, which increases with m (Fig. 3). This transition differs from standard equilibrium phase transitions known in physics, which are most often driven by the competition between energy and entropy. Here, the transition is driven by a competition between the collective and individual components of the agents' dynamics. The unsatisfactory global state of the city can be interpreted, from the economic point of view, as an effect of externalities: By moving to increase its utility, an agent may decrease other agents' utilities without taking this into account. From a standard interpretation in terms of Pigouvian tax (19), one expects that $\alpha = 1$ is necessary to reach the optimal state because by definition this value internalizes all the externalities the agent causes to the others when moving. Our results show that the optimal state is maintained until much lower tax values are

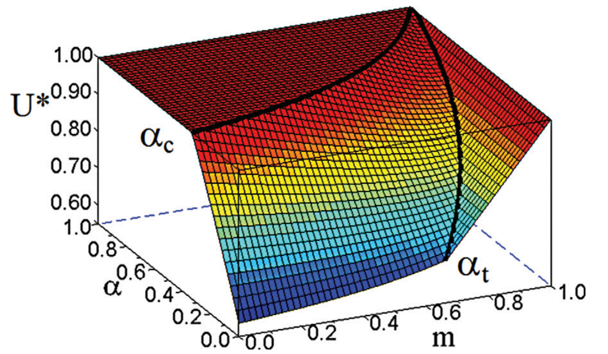

Fig. 3. Phase diagram of the global utility as a function of the cooperativity α and the asymmetry m, at $T \to 0$ and $\rho_0 = 1/2$. The average utility per agent $U^* = U/(\rho_0 HQ)$ is calculated by maximizing the potential $F(x)$ for the peaked utility shown in Fig. 2 (see *SI Appendix*). The plateau at high values of α corresponds to the mixed phase of optimal utility, which is separated from the segregated state by a phase transition arising at $\alpha_c = 1/(3-2m)$. The overall picture is qualitatively unchanged for low but finite values of the temperature, (see *SI Appendix*).

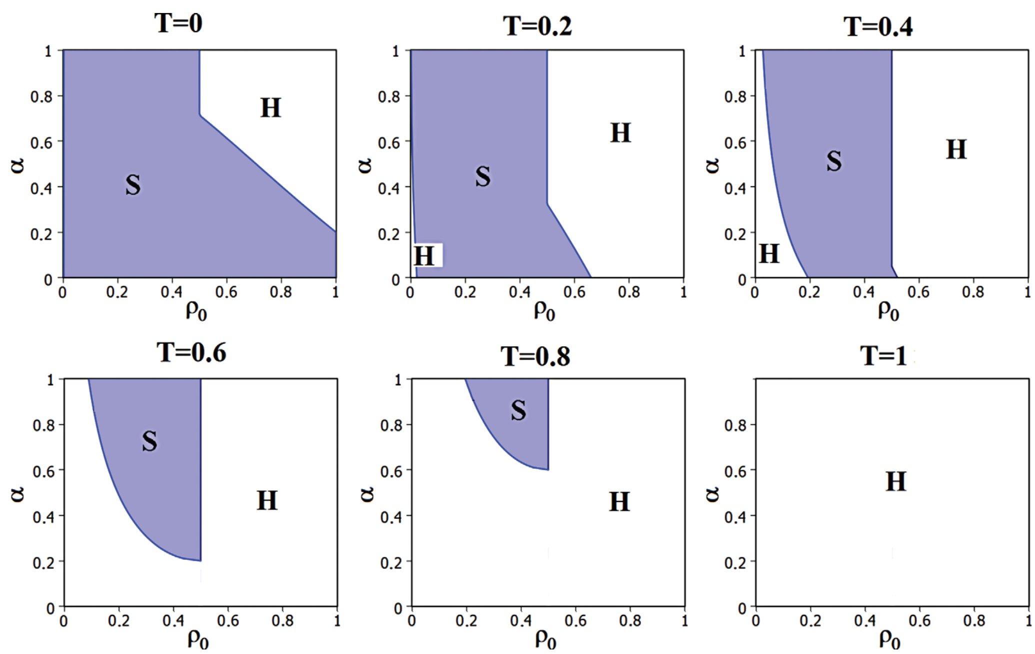

Fig. 4. Phase diagrams for the asymmetrically peaked individual utility (Fig. 2, with $m = 0.8$) for different values of T. Increasing the temperature T tends to favor homogeneous states. For small but finite temperatures (roughly $T < 0.2$), the phase diagram is modified only for extremal values of ρ_0, as expected from the entropic term $Ts(\rho) = -T\rho\ln\rho - T(1-\rho)\ln(1-\rho)$. As T is increased, the whole diagram is affected by the entropic term. Compared with the $T = 0$ case, the main change is the appearance of a second homogeneous phase for $\rho_0 < 1/2$. Although for $\rho_0 > 1/2$ homogeneity corresponds to the optimal choice for the agents, for $\rho_0 < 1/2$, collective utility is not maximized in a homogeneous city. The city is homogeneous by noise, not by choice. Note that an increase in α tends to reduce this domain, whereas it tends to increase the homogeneous domain for $\rho_0 > 1/2$.

reached (for example, $\alpha_c = 1/3$ at $m = 0$), a surprising result which deserves further analysis. Another interesting effect is observed for $m > 2/3$ (Fig. 3). Introducing a small tax has no effect on the overall satisfaction, the utility remaining constant until a threshold level is attained at $\alpha_t = (3m - 2)/(6 - 5m)$.

We focused up to now on the zero temperature limit. For low temperatures, the main qualitative conclusions are not modified, as the phase diagram is modified only for extremal values of ρ_0 by entropic contributions. At higher temperatures, the city tends to become homogeneous as the effect of "noise" (i.e., of the features that are not described in the model) dominates over the utility associated with the densities of the blocks (see Fig. 4).

Discussion

In the limit $\alpha = 0$, our model becomes similar to Schelling's segregation model (8), with two main differences: the existence of two types of agents of two colors and the definition of the agent's neighborhoods. We now show that these additional features do not introduce any essential effect.

Let us start by introducing two types of agents with different "colors" (such as red and green). Simple calculations (see *SI Appendix*) show that for two species that only care about the density of neighbors of their own color, the block potential Eq. **6** becomes

$$\begin{aligned} f(\rho_R, \rho_G) &= -T\rho_R \ln \rho_R - T\rho_G \ln \rho_G \\ &\quad - T(1-\rho_R-\rho_G)\ln(1-\rho_R-\rho_G) \\ &\quad + \alpha\Big[\rho_R u_R(\rho_R) + \rho_G u_G(\rho_G)\Big] \\ &\quad + (1-\alpha)\Big[\int_0^{\rho_R} u_R(\rho')d\rho' + \int_0^{\rho_G} u_G(\rho')d\rho'\Big], \end{aligned}$$

with straightforward notations (for example, $u_R(\rho_R)$ represents the utility of a red agent in a block with a density ρ_R of red agents). In the more general case of utility functions depending on both the density of similar and dissimilar neighbors, it is also possible to derive a block potential if the utility functions verify a symmetry constraint. This constraint is not very restrictive, in the sense that no qualitative feature of the model is lost when one restrains the study to utilities that verify it (see *SI Appendix*).

Finding the equilibrium configurations amounts to finding the set $\{\rho_{qR}, \rho_{qG}\}$, which maximizes the potential $F(x) = \sum_q f(\rho_{qR}, \rho_{qG})$ with the constraints $\sum_q \rho_{qR} = Q\rho_{0R}$ and $\sum_q \rho_{qG} = Q\rho_{0G}$, where ρ_{0G} and ρ_{0R} represent, respectively, the overall concentration of green and red agents.

Because of the spatial constraints (the densities of red and green agents in each block q must verify $\rho_{qR} + \rho_{qG} \le 1$), the "two populations" model cannot formally be reduced to two independent "one population" models. However, the stationary states can still be easily computed. Let us focus once again on the $T \to 0$ limit and suppose, for example, that $\rho_{0R} = \rho_{0G} = \rho_0/2$. The stationary state depends once again on the values of ρ_0, m, and α. For low values of α, it can be shown that the system settles in segregated states where each block contains only one kind of agent with a density ρ_0 (see Fig. 5*A*). For $\alpha \ge \alpha_c$, the system settles in mixed states where the density of a group in a block is either 0 or 1/2 (see Fig. 5*B*). The reader is referred to *SI Appendix* for more details.

We now turn to the difference in agents' neighborhoods. In Schelling's original model, agents' neighbors are defined as their eight nearest neighbors. Our model considers instead predefined blocks of common neighbors. First, it should be noted that there is no decisive argument in favor of either neighborhood definition in terms of the realism of the description of real social neighborhoods. Second, we note that introducing blocks allows for an analytical solution for arbitrary utility functions. This contrasts with the nearest neighbor case, where the best analytical approach solves only a modified model that abandons the individual point of view and is limited to a specific utility function (11). Finally, the simulations presented in Fig. 5 show that the transition from segregated to mixed states is not affected by the choice of the neighborhood's definition. We conclude that the block description is more adapted to this kind of simple modeling, which aims at showing stylized facts as segregation transitions.

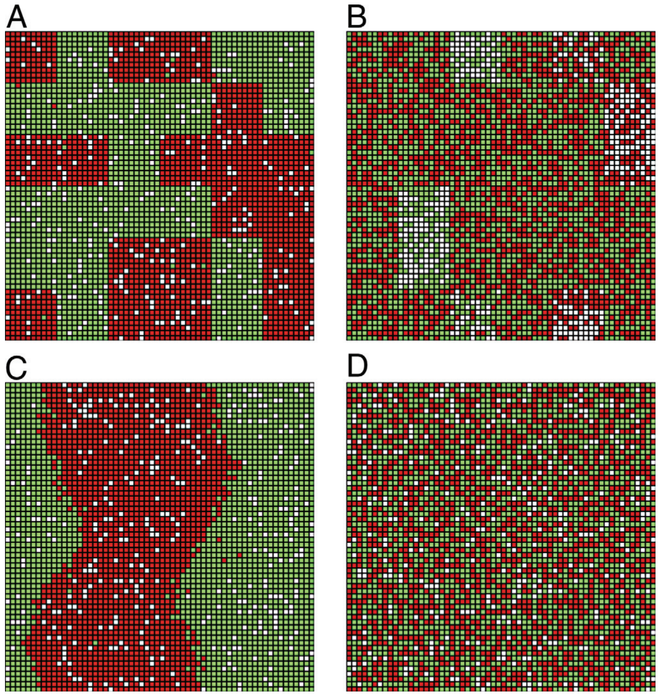

Fig. 5. Stationary configurations obtained by simulating the evolution of a city inhabited by an equal number of red and green agents whose preferences are given by the asymmetrically peaked utility function ($m = 0.5$). The rate of vacant cells (in white) is fixed to 10%. (*A* and *B*) The city is divided into blocks of size $H = 100$. In accordance with the analytic model, a segregated configuration is obtained when $\alpha = 0$ (*A*) and a more homogeneous configuration is obtained for $\alpha = 1$ (*B*). (*C* and *D*) The utility of an agent depends on the local density of similar neighbors computed on the $H = 108$ nearest cells. Although of different topological nature, a segregated configuration is still obtained for $\alpha = 0$ (*C*) and a homogeneous configuration is still obtained for $\alpha = 1$ (*D*). In all those simulations, we take $T = 0.1$. The small amount of noise hence generated, although not changing the nature of the stationary states compared with the case $T \to 0$, conveniently reduces the time of convergence of the system.

Our simple model raises a number of interesting questions about collective or individual points of view. In the purely collective case ($\alpha = 1$), the stationary state corresponds to the maximization of the average utility, in analogy to the minimization of energy in physics. In the opposite case ($\alpha = 0$), the stationary state strongly differs from the simple collection of individual optima (20): The optimization strategy based on purely individual dynamics fails, illustrating the unexpected links between micromotives and macrobehavior (21). However, the emergent collective state can be efficiently captured by the maximization of the link function $\ell(\rho)$ given in Eq. **5**, up to constraints in the overall town density. This function intimately connects the individual and global points of view. First, it depends only on the global town configuration (given by the ρ_q), allowing a relatively simple calculation of the equilibrium. At the same time, it can be interpreted as the sum of the individual marginal utilities gained by agents as they progressively fill the city after leaving a reservoir of zero utility. In the stationary state, a maximal value of the potential L is reached. Therefore, no agent can increase its utility by moving (because $\Delta u = \Delta L$), consistent with the economists' definition of a Nash equilibrium.

Equilibrium statistical mechanics has developed powerful tools to link the microscopic and macroscopic levels. These tools are limited to physical systems, where dynamics are governed by a global quantity such as the total energy. By introducing a link function, analogous to state functions in thermodynamics or potential functions in game theory (22), we have extended the framework of statistical mechanics to a Schelling-like model. Such an approach paves the way to analytical treatments of a much wider class of systems, where dynamics are governed by individual strategies.

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SUPPORTING INFORMATION

1 Phase separation

Focusing on the large H case, the problem gets back to finding the set $\{\rho_q\}$ which maximize the potential $F(x) = H \sum_q f(\rho_q)$ with the constraint $\sum_q \rho_q$ fixed. We are interested to know whether the stationary state is statistically homogeneous or inhomogeneous. Following standard physics textbooks methods, the homogeneous state at density ρ_0 is unstable against a phase separation if there exists two densities ρ_1 and ρ_2 such that

$$\gamma f(\rho_1) + (1-\gamma) f(\rho_2) > f(\rho_0). \tag{1}$$

The parameter γ ($0 < \gamma < 1$) corresponds to the fraction of blocks that would have a density ρ_1 in the diphasic state, while a fraction $1-\gamma$ would have a density ρ_2. This condition simply means that the value of the sum $\sum_q f(\rho_q)$ is higher for the diphasic state than for the homogeneous state, so that the diphasic state has a much larger probability to occur. Geometrically, the inequality (1) corresponds to requiring that $f(\rho)$ is a non-concave function of ρ. The values of ρ_1 and ρ_2 are obtained by maximizing $\gamma f(\rho_1') + (1-\gamma) f(\rho_2')$ over all possible values of ρ_1' and ρ_2', with γ determined by the mass conservation $\gamma \rho_1' + (1-\gamma)\rho_2' = \rho_0$.

Further, the equilibrium coexistence points to a given temperature can be determined by a double tangent method where the equilibrium densities of the individual phase fall on the same tangent line of $f(\rho)$. The first derivatives of f are equivalent at these two densities and also equal to the slope connecting these two points, *ie*,

$$\frac{f(\rho_2) - f(\rho_1)}{\rho_2 - \rho_1} = f'(\rho_1) \tag{2}$$

$$\frac{f(\rho_2) - f(\rho_1)}{\rho_2 - \rho_1} = f'(\rho_2) \tag{3}$$

For the computation of functions depending on the global state of the city such as the normalized collective utility $U^*(x) = U(x)/\sum_q n_q$, two cases have to be distinguished

- The case when there is one phase of density ρ_0. In this case, the utility of each agent is equal to the normalized collective utility :

$$U^*(x) = u(\rho_0) \tag{4}$$

- The case when there are two phases of densities ρ_1 and ρ_2. In this case, the normalized collective utility can be written as

$$U^*(x) = \gamma \frac{\rho_1}{\rho_0} u(\rho_1) + (1-\gamma) \frac{\rho_2}{\rho_0} u(\rho_2) \tag{5}$$

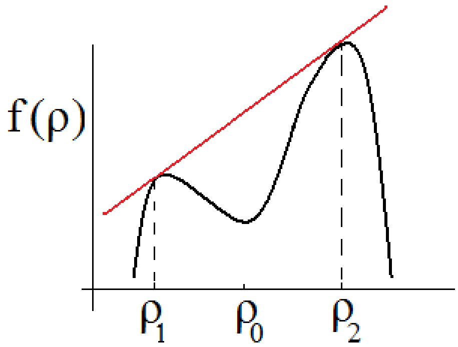

Figure 1: **Phase separation**. The system of density ρ_0 splits into two phases of densities ρ_1 and ρ_2 if it increases its potential. The standard double tangent construction determines the densities of the two phases at equilibrium.

with the conservation of the number of agents providing the value of the fraction $\gamma = (\rho_2 - \rho_0)/(\rho_2 - \rho_1)$ of blocks of density ρ_1.

2 Asymmetrically peaked utility function

The specific form of the utility function is an input of the model, and it can be postulated on a phenomenological basis, or rely on a theory of the interactions among agents. To illustrate the influence of the parameter α, we choose to work with the asymmetrically peaked utility function defined as:

$$\begin{aligned} u(\rho) &= 2\rho && \text{if } \rho \leq \frac{1}{2} \\ u(\rho) &= m + 2(1-m)(1-\rho) && \text{if } \rho > \frac{1}{2} \end{aligned}$$

where $m < 1$ is a real parameter.

It is straightforward to verify that the function $f(\rho)$ reads for $\rho \leq 1/2$

$$f(\rho) = -T\big(\rho \ln \rho + (1-\rho)\ln(1-\rho)\big) + (1+\alpha)\rho^2 \tag{6}$$

and similarly, for $\rho > 1/2$

$$f(\rho) = -T\big(\rho \ln \rho + (1-\rho)\ln(1-\rho)\big) - (1+\alpha)(1-m)\rho^2 + (2-m)\rho - (1-\alpha)(2-m)/4 \tag{7}$$

2.1 Limiting case $T \to 0$

Let us first consider the limiting case $T \to 0$. From the above expression of $f(\rho)$, it turns out that $f(\rho)$ is convex for $0 < \rho < 1/2$ and concave for $1/2 < \rho < 1$, as $1 - m > 0$. Thanks to Fig. 2, it is pretty clear that there exists $\rho_2(\alpha, m) > 1/2$ such that a phase of mean density ρ_0 is stable if $\rho_0 \geq \rho_2(\alpha, m)$. In the opposite

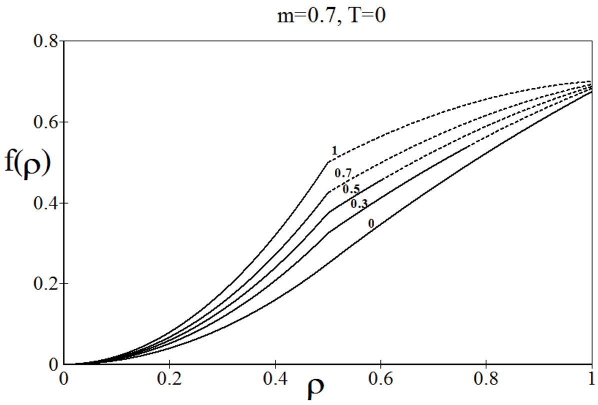

Figure 2: Graphic representation of the f function for $m = 0.7$, $T = 0$ and $\alpha = 0, 0.3, 0.5, 0.7, 1$. The dash lines represent the part of the curves merging with their concave hulls. The solid line hence corresponds to the range of mean densities ρ_0 for which there is phase separation.

case, a phase separation occurs, and the densities ρ_1 and ρ_2 can be computed as previously explained.

However, in the limit $T = 0$, the line joining ρ_1 and ρ_2 does not correspond to a double tangent. Due to the convexity of f on $[0, 1/2]$, one has $\rho_1 = 0$ and $f'(\rho_1) = 0$. To determine ρ_2, we first assume that $1/2 < \rho_2 < 1$, so that the line joining $\rho_1 = 0$ to ρ_2 is a tangent to f at ρ_2, which is expressed as:

$$f'(\rho_2) = \frac{1}{\rho_2}\big(f(\rho_2) - f(0)\big), \tag{8}$$

yielding

$$\rho_2 = \frac{1}{2}\sqrt{\frac{1-\alpha}{1+\alpha}\;\frac{2-m}{1-m}}. \tag{9}$$

From Eq. (9), we find that ρ_2 is in the range $1/2 < \rho_2 < 1$ if (and only if) the following condition is satisfied:

$$\frac{3m-2}{6-5m} = \alpha_t(m) < \alpha < \alpha_c(m) = \frac{1}{3-2m}. \tag{10}$$

Hence for $\alpha \geq \alpha_c(m)$, ρ_2 sticks to the value $\rho_2 = 1/2$. Similarly, for $\alpha \leq \alpha_t(m)$, one has $\rho_2 = 1$. These results are illustrated on Fig 3. The dependency of the outcome with the mean density of agents is quite simple. For $\rho_0 < \rho_2(\alpha, m)$, two kinds of blocks coexist in the stationary states: empty blocks and blocks of density ρ_2. The collective utility can then be written as

$$U^*(x) = u(\rho_2) = \begin{cases} m \text{ if } \alpha \leq \alpha_t \\ 2 - m - \sqrt{\frac{1-\alpha}{1+\alpha}(2-m)(1-m)} \text{ if } \alpha_t \leq \alpha \leq \alpha_c \\ 1 \text{ if } \alpha \geq \alpha_c \end{cases} \tag{11}$$

This expression clearly increases with α, as expected.

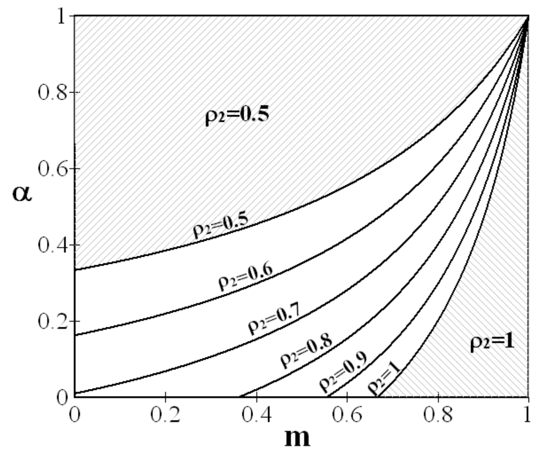

Figure 3: **Values of $\rho_2(\alpha, m)$ at $T = 0$.** If $\rho_0 < \rho_2(\alpha, m)$, the system of density ρ_0 splits into two phases of densities $\rho_1 = 0$ and $\rho_2 = \rho_2(\alpha, m)$ to increase the value of the potential $F(x)$. Otherwise, the equilibrium corresponds to the homogeneous phase of density ρ_0.

In the opposite case for which $\rho_0 \geq \rho_2(\alpha, m)$, the density of the blocks in the stationary states is homogeneous and the collective utility is then

$$U^*(x) = u(\rho_0) = 2 - m - 2(1 - m)\rho_0 \tag{12}$$

Notice furthermore that the independence of collective utility with the tax parameter α observed on Fig 3 of the article for $\alpha > \alpha_c$ and $\alpha < \alpha_t$ correspond to domains for which the density ρ_2 has reached a saturation value (respectively 1 or 1/2).

The phase diagrams presented on Fig. 4 give a more precise idea of the influence of the parameter ρ_0 over the different phases in the stationary states.

2.2 Finite temperatures

Finally, we turn to the analysis of the model for finite values of T and show that the behavior of the model remains qualitatively similar to that obtained previously in the $T \to 0$ limit. The high T case is the simplest to analyze. For $2T/(1+\alpha) \geq \max_{[0,1]} \big(4\rho(1-\rho)\big) = 1$, f is concave on the two intervals $[0, 1/2[$ and $]1/2, 1]$ where it is regular. One can moreover verify that at the singular point $\rho = 1/2$, $f'(1/2^+) > f'(1/2^-)$, which ensures that f is concave on the whole interval $[0, 1]$. Hence for $T/(1+\alpha) > 1/2$, there is a single phase of density ρ_0.

In the opposite case $0 < T/(1 + \alpha) < 1/2$, the analysis is somewhat similar to the zero T limit. The function f is convex on the interval

$$\frac{1}{2}\left(1 - \sqrt{1 - \frac{2T}{1+\alpha}}\right) < \rho < \frac{1}{2} \tag{13}$$

and concave on the complementary interval. As $f(\rho)$ has an infinite slope in

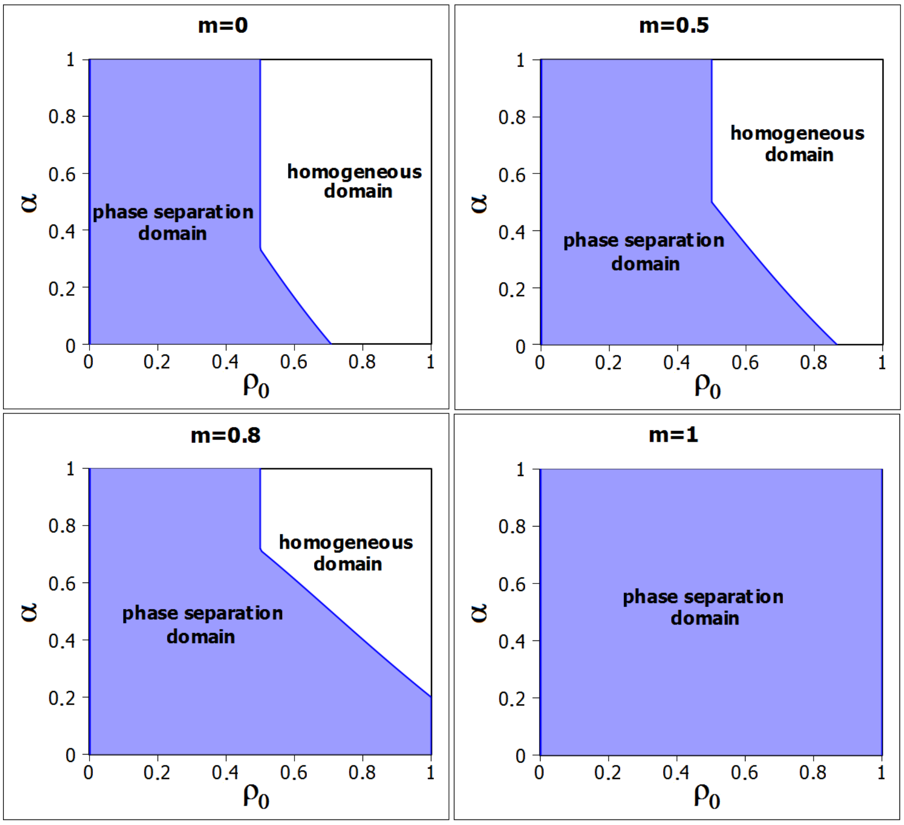

Figure 4: **Phase diagrams** at $T = 0$ for different values of m. For $\rho_0 > 1/2$, phase separation is always a disadvantage in terms of collective utility. The homogeneous phase, which maximizes the collective utility is stable from a certain value of α. For $\rho_0 \leq 1/2$, the collective utility is maximal for a separation into two phases of densities $\rho_1 = 0$ and $\rho_2 = 1/2$, a separation obtained in the stationary states when $\alpha > 1/(3-2m)$. For lower values of the tax, phase separation is a disadvantage.

$\rho = 0$ and $\rho = 1$, the densities ρ_1 and ρ_2 satisfy $0 < \rho_1 < 1/2$ and $1/2 \leq \rho_2 < 1$. Assuming that $\rho_2 = 1/2$, the density ρ_1 is given by the implicit expression

$$\frac{(1-2\rho_1)^2}{\ln\left(4\rho_1(1-\rho_1)\right)} = -\frac{2T}{1+\alpha}. \tag{14}$$

Then the assumption $\rho_2 = 1/2$ is consistent as long as $f'(1/2^+) \geq (f(1/2) - f(\rho_1))/(1/2 - \rho_1)$, which can be rewritten as

$$\varphi(\rho_1) \geq \frac{1-\alpha(3-2m)}{1+\alpha} \tag{15}$$

where the function φ is defined by

$$\varphi(\rho) = 4\rho - 1 + (1-2\rho)^2 \frac{\ln\rho - \ln(1-\rho)}{\ln(4\rho(1-\rho))}. \tag{16}$$

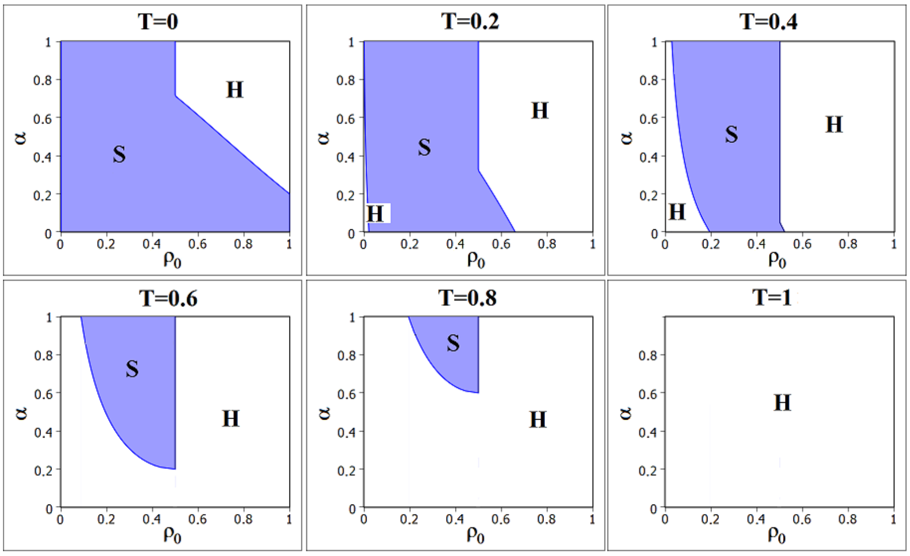

Figure 5: **Phase diagrams** at $m = 0.8$ for different values of T. Increasing the "temperature" T tends to favour homogeneous states. For $T \to 0$, the phase diagram is affected only for extremal values of ρ_0, as can be expected from the entropic term $Ts(\rho) = -T\rho \ln \rho - T(1-\rho)\ln(1-\rho)$. As T is increased, all the diagram is affected by the entropic term. Compared to the $T = 0$ case, the main change at low T is the onset of a second homogeneous phase for $\rho_0 < 1/2$. But whereas for $\rho_0 > 1/2$ homogeneity corresponds to the best interest of the agents, for $\rho_0 < 1/2$, collective utility is not maximized in an homogeneous city. This homogeneous domain is here purely induced by noise. Note that an increase in α tends to reduce this domain while it tends to increase the homogeneous domain for $\rho_0 > 1/2$.

Note that the inequality (15) is automatically verified if $\alpha \geq 1/(3-2m)$, as the function φ is positive. If the inequality (15) is not satisfied, then $\rho_2 > 1/2$, and the values of ρ_1 and ρ_2 are solutions of two coupled non-linear equations, that can be solved numerically.

The phase diagrams presented on Fig 5 give a idea of the influence of the "temperature" T over the stationary states of the system.

3 Model with two types of agents - one variable utility functions

We present in this section extended results of a model with two types of agents in the case where the utility of the agents depends only on the number of similar neighbors. Section 4 present some results for the case when the utility of the agents depends both on the number of similar and dissimilar neighbors.

3.1 Bases of the model

3.1.1 Notations

In this section, describing a city inhabited by two types of agents (that we refer to as red and green agents), we will note:

Q the number of blocks the city is divided in, each block being composed of H cells;

x a configuration of the city, corresponding to the knowledge of the state (empty, red or green) of each cell;

$n_{qr}(x)$ and $n_{qg}(x)$ the numbers of red and green agents living in the block q;

$u(n_{qr}/H)$ (resp $u(n_{qg}/H)$) the utility of a red (resp green) agent living in block q, with $u(0) = 0$ by convention;

$N_0 = \sum_q n_q \leq QH$ the total number of agents;

$N_R = \sum_q n_{qr}$ the total number of red agents (idem for the green ones);

$U(x) = \sum_q (n_{qr} u(n_{qr}/H) + n_{qg} u(n_{qg}/H))$ the total utility in configuration x;

$L(x) = \sum_q \left(\sum_{m=0}^{n_{qr}} u(m/H) + \sum_{m=0}^{n_{qg}} u(m/H) \right)$ the value of the "linking function" in configuration x.

$0 \leq \alpha \leq 1$ the tax parameter.

3.1.2 Dynamic rule

At each iteration, one picks at random an agent and a vacant cell. The agent moves in this empty cell with a probability

$$Pr\{move\} = \frac{1}{1 + e^{-(\Delta u + \alpha(\Delta U - \Delta u))/T}} = \frac{1}{1 + e^{-((1-\alpha)\Delta u + \alpha \Delta U)/T}} \tag{17}$$

where Δu is the variation of utility which the chosen agent can achieve by moving, ΔU is the variation of global utility which would result from this same move and $0 \leq \alpha \leq 1$ is an "altruism" parameter (for $\alpha = 0$ the move only depends on the egoistic interest of the agent, for $\alpha = 1$ it only depends on the collective interest).

3.1.3 A potential function

Let us define two states x and y as *immediately communicating states* (ICS) if we can switch from state x to state y by moving one single agent. Whatever the form of the utility function u, one has for every move $\Delta u = \Delta L$. The transition probability from a configuration x to a configuration y in one iteration can thus be written:

$$\begin{aligned} P_{xy} &= \gamma_{xy} \frac{1}{1 + e^{-((1-\alpha)(L(y)-L(x)) + \alpha(U(y)-U(x)))/T}} \\ &= \gamma_{xy} \frac{e^{((1-\alpha)L(y)+\alpha U(y))/T}}{e^{((1-\alpha)L(x)+\alpha U(x))/T} + e^{((1-\alpha)L(y)+\alpha U(y))/T}} \end{aligned}$$

where γ_{xy} takes into account the probability to pick the right agent and the right vacant cell that allow to pass from x to y:

$$\gamma_{xy} = \frac{1}{N_0(QH - N_0)} \quad \text{if } x \text{ and } y \text{ are ICS,} \tag{18}$$

$$\gamma_{xy} = 0 \quad \text{if } x \text{ and } y \text{ are not ICS.} \tag{19}$$

Since the function

$$\Pi(x) = \frac{e^{(1-\alpha)L(x)+\alpha U(x)}}{\sum_z e^{(1-\alpha)L(z)+\alpha U(z)}} \tag{20}$$

is the unique normalized function that verifies for all x and y the detailed balance:

$$\Pi(x)P_{xy} = \Pi(y)P_{yx} \tag{21}$$

one can identify Π as the stationary distribution function.

There is $\frac{H!}{n_R!n_G!(H-n_R-n_G)!}$ ways of ordering n_R undifferentiated red agents and n_G undifferentiated green agents in H cells. Indeed, there is $\frac{H!}{(n_R+n_G)!(H-n_R-n_G)!}$ ways of placing the vacant cells and $\frac{(n_R+n_G)!}{n_R!n_G!}$ ways of placing the agents' colors. So one can compute the stationary distribution function for the coarse-grained states $\{\rho_q\}$:

$$\Pi(\{n_q\}) = \frac{1}{Z}\prod_q \frac{H!}{n_R!n_G!(H-n_R-n_G)!} e^{\left((1-\alpha)L(x)+\alpha U(x)\right)/T} \tag{22}$$

$$= \frac{1}{Z} e^{H/T\sum_q f(n_q,T,H)} \tag{23}$$

where

$$\begin{aligned} f(n_R, n_G, T, H) &= -\frac{T}{H}\ln\Big(\frac{n_R!n_G!(H-n_R-n_G)!}{H!}\Big) \\ &+ \alpha\frac{n_R}{H}u_R\Big(\frac{n_R}{H}\Big) + \alpha\frac{n_G}{H}u_G\Big(\frac{n_G}{H}\Big) \\ &+ (1-\alpha)\frac{1}{H}\sum_{m=0}^{n_R} u_R\Big(\frac{m}{H}\Big) + (1-\alpha)\frac{1}{H}\sum_{m=0}^{n_G} u_G\Big(\frac{m}{H}\Big) \end{aligned}$$

The configurations that maximize the potential $F(x) = \sum_q f(n_R, n_G, T)$ are the more probable to come up. In the limit $H/T \to \infty$, these configurations are even the only ones that will appear in the stationary states (since $\Pi(x)/\Pi(y) = e^{H/T(F(x)-F(y))} \to 0$ for $F(x) - F(y) < 0$ and $H/T \to \infty$).

3.1.4 Continuous limit

In the limit $H \to \infty$, by keeping constant the mean density $\rho_0 = N_0/H$ and the density of each block $\rho_q = n_q/H$ (ρ_q hence becoming a continuous variable), one has thanks to Stirling's formula:

$$\ln\Big(\frac{n_R!n_G!(H-n_R-n_G)!}{H!}\Big) \simeq H\Big(\rho_{qR}\ln\rho_{qR} + \rho_{qG}\ln\rho_{qG} + (1-\rho_{qR}-\rho_{qG})\ln(1-\rho_{qR}-\rho_{qG})\Big)$$

and the stationary distribution can be written as:

$$\Pi(\{\rho_q\}) = \frac{1}{Z} \prod_q e^{H/T f(\rho_{qR}, \rho_{qG}, T)} \tag{24}$$

where the "block-potential" is

$$\begin{aligned} f(\rho_R, \rho_G, T) &= -T\rho_R \ln \rho_R - T\rho_G \ln \rho_G - T(1-\rho_R-\rho_G)\ln(1-\rho_R-\rho_G) \\ &+ \alpha\rho_R u_R(\rho_R) + \alpha\rho_G u_G(\rho_G) \\ &+ (1-\alpha)\int_0^{\rho_R} u_R(\rho')d\rho' + (1-\alpha)\int_0^{\rho_G} u_G(\rho')d\rho' \end{aligned}$$

The problem hence gets back to find the set $\{\rho_{qR}, \rho_{qG}\}$ which maximizes the potential $F = \sum_q f(\rho_{qR}, \rho_{qG}, T)$ with the constraints $\sum_q \rho_{qR} = Q\rho_{0R}$ and $\sum_q \rho_{qG} = Q\rho_{0G}$.

Comparing this result to the result of the one population model, the 'two populations model' for $T = 0$ is similar to the sum of two 'one population models', one for each color. The only difference is introduced by the spatial constraint: the densities of red and green agents in each block q must verify $\rho_{qR} + \rho_{qG} \leq 1$. At non-zero temperature the $-T(1-\rho_R-\rho_G)\ln(1-\rho_R-\rho_G)$ term links both populations.

3.2 Homogeneous-inhomogeneous transitions

The homogeneous phase may be unstable with respect to phase separation.
Let us split the system into two phases of densities $\rho_1 = (\rho_{1R}, \rho_{1G})$ and $\rho_2 = (\rho_{2R}, \rho_{2G})$. The constraint that the overall densities of particles/agents are $\rho_0 = (\rho_{0R}, \rho_{0G})$ is expressed by the lever rule:

$$\begin{cases} Q_1 + Q_2 &= Q \\ Q_1\rho_1 + Q_2\rho_2 &= Q\rho_0 \end{cases}$$

where Q_1 and Q_2 are respectively the number of blocks of density ρ_1 and ρ_2. The homogeneous phase is stable against phase separation if for all ρ_1 and ρ_2

$$Q_1 f(\rho_1) + Q_2 f(\rho_2) < Q f(\rho_0) \tag{25}$$

Geometrically, this inequality corresponds to requiring that $f(\rho)$ is a concave function.
When the concavity requirement is violated, phase separation will occur for certain values of ρ_0. The equilibrium densities ρ_1 and ρ_2 are such that the line that joins the points $(\rho_1,\ f(\rho_1))$ and $(\rho_2, f(\rho_2))$ is part of the concave hull of the function.
In the 2 populations model there is a possibility that the system is split into 3 phases of densities $\rho_1 = (\rho_{1R}, \rho_{1G})$, $\rho_2 = (\rho_{2R}, \rho_{2G})$ and $\rho_3 = (\rho_{3R}, \rho_{3G})$. The constraint that the overall densities of particles/agents are $\rho_0 = (\rho_{0R}, \rho_{0G})$ is now:

$$\begin{cases} Q_1 + Q_2 + Q_3 &= Q \\ Q_1\rho_{1R} + Q_2\rho_{2R} + Q_3\rho_{3R} &= Q\rho_{0R} \\ Q_1\rho_{1G} + Q_2\rho_{2G} + Q_3\rho_{3G} &= Q\rho_{0G} \end{cases}$$

where Q_1, Q_2 and Q_3 are respectively the number of blocks of density ρ_1, ρ_2 and ρ_3.
And the equilibrium densities ρ_1, ρ_2 and ρ_3 are now such that the plane that joins the points $(\rho_1, f(\rho_1))$, $(\rho_2,f(\rho_2))$ and $(\rho_3,f(\rho_3))$ is part of the concave hull of the function.
For some values of the parameters there may even be 4 points of the same plane belonging to the f function and its concave hull. In this case there will be a continuum of possible values of Q_1, Q_2, Q_3 and Q_4 verifying the global density constraints.

3.3 With a peaked utility function

Expression of the f function

Let us consider for both populations the asymmetrically peaked utility function defined for $m < 1$ as:

$$
\begin{aligned}
u(\rho) &= 2\rho \quad \text{if } \rho \leq 0.5 \\
u(\rho) &= m + 2(1-m)(1-\rho) \quad \text{if } \rho > 0.5
\end{aligned}
$$

For $\rho_R \leq 0.5$ and $\rho_G \leq 0.5$, the f function is:

$$
\begin{aligned}
f(r,g) &= -T\big(r\ln r - g\ln g - (1-r-g)\ln(1-r-g)\big) + (1+\alpha)(r^2+g^2) \\
\frac{\partial f}{\partial r}(r,g) &= -T\big(\ln r - \ln(1-r-g)\big) + 2(1+\alpha)r \\
\frac{\partial^2 f}{\partial r^2}(r,g) &= -T/r - T/(1-r) + 2(1+\alpha)
\end{aligned}
$$

(Partial derivatives relative to g are obtained by replacing $g \leftrightarrow r$)

For $\rho_R > 0.5$ and $\rho_G \leq 0.5$:

$$
\begin{aligned}
f(r,g) &= -T\big(r\ln r + g\ln g + (1-r-g)\ln(1-r-g)\big) - (1+\alpha)(1-m)r^2 + (2-m)r \\
&- (1-\alpha)(2-m)/4 + (1+\alpha)g^2 \\
\frac{\partial f}{\partial r}(r,g) &= -T\big(\ln r - \ln(1-r-g)\big) - 2(1+\alpha)(1-m)r - (2-m) \\
\frac{\partial^2 f}{\partial r^2}(r,g) &= -T/r - T/(1-r-g) - 2(1+\alpha)(1-m) \\
\frac{\partial f}{\partial g}(r,g) &= -T\big(\ln g - \ln(1-r-g)\big) + 2(1+\alpha)r \\
\frac{\partial^2 f}{\partial r^2}(r,g) &= -T/g - T/(1-r-g) + 2(1+\alpha)
\end{aligned}
$$

The situation $\rho_R \leq 0.5$ and $\rho_G > 0.5$ can be obtained by replacing $g \leftrightarrow r$ in the previous paragraph.

For $T = 0$

f is concave in ρ_R and ρ_G for ρ_R and $\rho_G \leq 0.5$. For $\rho_R > 0.5$ and $\rho_G \leq 0.5$, f is concave in ρ_R and convex in ρ_G (and conversely for $\rho_R \leq 0.5$ and $\rho_G > 0.5$,

f is concave in ρ_G and convex in ρ_R).

The concave hull of the function has a different form for different values of the parameters α and m:

- for $\alpha \geq \frac{1}{3-2m}$ the points $\left(0, \frac{1}{2}, f(0, \frac{1}{2})\right)$ and $\left(\frac{1}{2}, 0, f(\frac{1}{2}, 0)\right)$ belong to the concave hull whereas for $\alpha \leq \frac{1}{3-2m}$ they are replaced by the points $\left(0, \rho_2, f(0, \rho_2)\right)$ and $\left(\rho_2, 0, f(\rho_2, 0)\right)$ with $\rho_2(\alpha, m) = \frac{1}{2}\sqrt{\frac{1-\alpha}{1+\alpha}\,\frac{2-m}{1-m}}$ (see the resolution of the one population model).
- for $\alpha \geq \frac{m}{4-3m}$ the point $\left(\frac{1}{2}, \frac{1}{2}, f(\frac{1}{2}, \frac{1}{2})\right)$ belongs to the concave hull whereas for $\alpha < \frac{m}{4-3m}$ it does not.

So there are three possible situations, shown on figure 6:

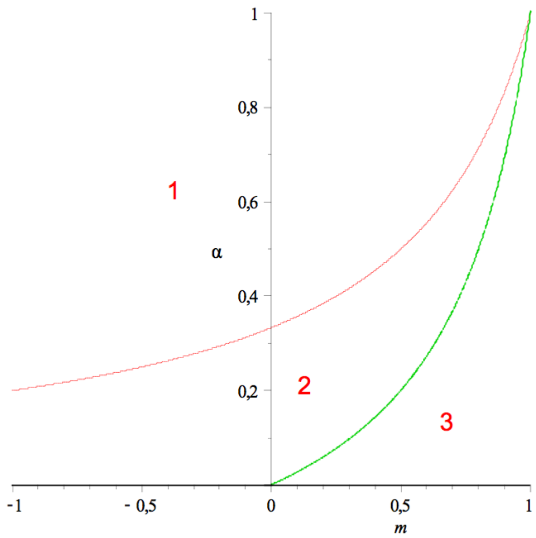

Figure 6: The domains of different concave hulls for different values of m and α

- $\alpha \geq \frac{1}{3-2m}$ (which will be case 1)
- $\frac{m}{4-3m} \leq \alpha \leq \frac{1}{3-2m}$ (case 2)
- $\alpha < \frac{m}{4-3m}$ (case 3)

Case 1

The number and composition of the phases depend on the global densities $\rho_0 = (\rho_{0R}, \rho_{0G})$ (see figure 7).

In part A of figure 7, the system separates into 3 or 4 phases of densities $(0,0)$, $(0, \frac{1}{2})$, $(\frac{1}{2}, 0)$ and $(\frac{1}{2}, \frac{1}{2})$ in respective quantities Q_1, Q_2, Q_3 and Q_4, which must verify

$$\begin{cases} Q_2 + Q_4 & = 2Q\rho_{0G} \\ Q_3 + Q_4 & = 2Q\rho_{0R} \\ Q_1 + Q_2 + Q_3 + Q_4 & = Q \end{cases}$$

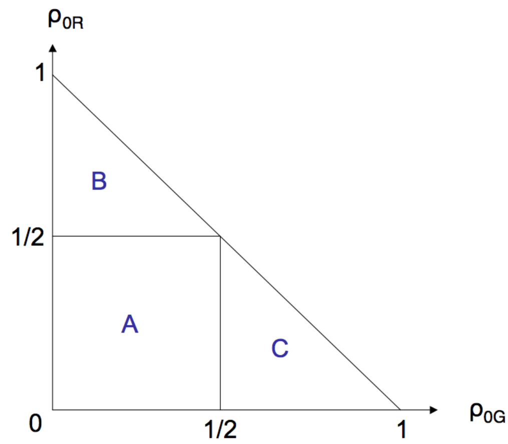

Figure 7: Domains of different phases for different global densities in case 1

The system can build either 3 or 4 phases because red and green agents do not "see" each other: their utility is maximal when half of the block is filled with agents of their color, the other half being either empty or filled with agents of the other color.
In part B, the system separates into 2 phases of densities $(\frac{\rho_{0R}-\rho_{0G}}{1-2\rho_{0G}}, 0)$ and $(\frac{1}{2}, \frac{1}{2})$ with respective weights $Q_1 = Q(1-2\rho_{0G})$ and $Q_2 = 2Q\rho_{0G}$.
And symmetrically in part C the system separates into 2 phases of densities $(0, \frac{\rho_{0G}-\rho_{0R}}{1-2\rho_{0R}})$ and $(\frac{1}{2}, \frac{1}{2})$ with respective weights $Q_1 = Q(1-2\rho_{0R})$ and $Q_2 = 2Q\rho_{0R}$.

Case 2

The number and composition of the phases depend again on the global densities as shown on figure 8.
In part A of figure 8, the system separates into 3 phases of densities $(0, 0)$, $(0, \rho_2(\alpha, m))$ and $(\rho_2(\alpha, m), 0)$ in respective quantities $Q_1 = Q(1 - \frac{\rho_{0R}+\rho_{0G}}{\rho_2})$, $Q_2 = Q\frac{\rho_{0G}}{\rho_2}$ and $Q_3 = Q\frac{\rho_{0R}}{\rho_2}$.
In part B the system is split into 3 phases of densities $(0, \rho_2)$, $(\rho_2, 0)$ and $(\frac{1}{2}, \frac{1}{2})$ in respective quantities $Q_1 = Q\frac{\rho_2(2\rho_{0G}-1)+\rho_{0R}-\rho_{0G}}{2\rho_2(\rho_2-1)}$, $Q_2 = Q\frac{\rho_2(2\rho_{0R}-1)+\rho_{0G}-\rho_{0R}}{2\rho_2(\rho_2-1)}$ and $Q_3 = Q\frac{\rho_2-\rho_{0R}-\rho_{0G}}{\rho_2-1}$.
In part C the phase decomposition is the same as in part B of case 1 and in part D it is the same as in part C of case 1.

Case 3

The different domains of phase decomposition are shown on figure 9.

In part A of figure 9 the system separates into 3 phases like in part A of case B.

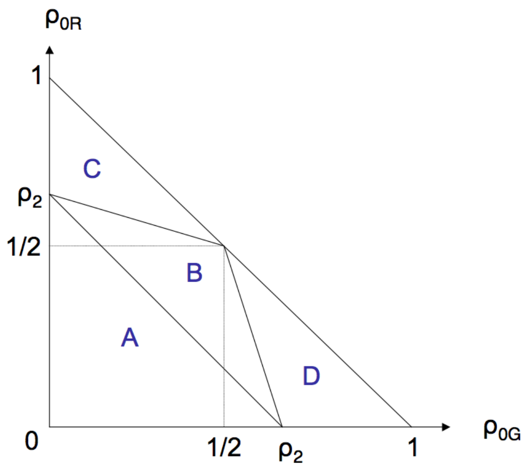

Figure 8: Domains of different phases for different global densities in case 2

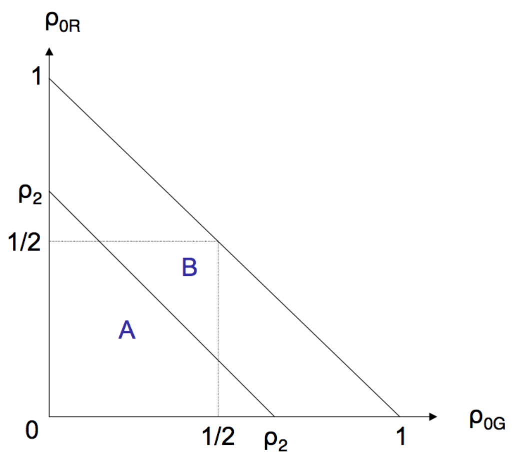

Figure 9: Domains of different phases for different global densities in case 3

In part B there are 2 phases of densities $(\rho_{0R} + \rho_{0G}, 0)$ and $(0, \rho_{0R} + \rho_{0G})$ in respective quantities $Q_1 = Q \frac{\rho_{0R}}{\rho_{0R}+\rho_{0G}}$ and $Q_2 = Q \frac{\rho_{0G}}{\rho_{0R}+\rho_{0G}}$.

Interpretation

At zero temperature, the two populations model is indeed very similar to the one population model: both populations of agents see each other only as occupied cells (the utility of a red agent does not depend on the number of green agents in the block). The main difference with two superimposed one population models lies in the fact that blocks with density $(\frac{1}{2}, \frac{1}{2})$ are full of agents with maximal utilities and cannot take in more agents, so that for certain values of the global densities, in cases 1 and 2, there is a phase of half-red, half-green blocks and another phase containing the excess of the more numerous type of agents with a higher density (and thus an inferior utility).

For $T > 0$

At high temperatures, the entropic term of the f function is the leading term, and as it is a concave one, the function is concave everywhere : for any density $\rho_0 = (\rho_{0R}, \rho_{0G})$, the system stays in an homogeneous phase because of the strong noise.
For intermediate values of the temperature, the system has a behavior between a noise driven one and the one it has at zero temperature, but we do not study it further here.

4 Model with two type of agents - two variables utility functions

We present here some preliminary results in the case when the utility functions may depend on the number of similar and dissimilar agent. The utility of a red (resp. green) agent living in block q is noted $u_R(n_{qR}/H, n_{qG}/H)$ (resp $u_G(n_{qG}/H, n_{qR}/H)$).

4.1 Definitions

Let $\mathbb{U}$ be the set of pairs of utility functions (u_R, u_G) that verify, for all $(n_R, n_G) \in E_H \equiv \{(n_R, n_G), 0 \leq n_R + n_G \leq H\}$, the following condition:

$$\begin{aligned} u_R(n_R/H, n_G/H) - u_R(n_R/H, n_G/H + 1/H) = \\ u_G(n_R/H, n_G/H) - u_G(n_R/H + 1/H, n_G/H) \end{aligned} \tag{26}$$

Condition (26) only imposes that if a block contains $n_R + 1$ red agents and $n_G + 1$ green agents, the utility gain a red agent would achieve if a green agent left must be the same as the utility gain a green would achieve if a red agent left. As we show below, this condition is not strongly restrictive from a theoretical viewpoint, which means that our approach can be applied to virtually all the usual utility functions.

Indeed, it is useful to remark that the set $\mathbb{U}$ is composed of the pairs of utility functions (u_R, u_G) which are written:

$$\begin{aligned} u_R(n_R/H, n_G/H) &= \xi_R(n_R/H) + \sum_{g=0}^{n_{qG}-1} \xi(n_{qR}/H, g/H) \\ u_G(n_R/H, n_G/H) &= \xi_G(n_G/H) + \sum_{r=0}^{n_{qR}-1} \xi(r/H, n_{qG}/H) \end{aligned} \tag{27}$$

where ξ_R and ξ_G are arbitrary functions of one variable and ξ an arbitrary function of two variables.

In the limit of a very low vacancy rate, there is no vacant cells in most of the blocks, *ie* in these blocks the relation $n_{qR} + n_{qG} = H$ holds. Hence, one only needs one parameter among (n_{qR}, n_{qG}) to define a utility function and considering for instance that an agent's utility only depends on his number of similar neighbors is sufficient to describe all possible cases. This can simply be done by taking $\xi \equiv 0$ in Eq. (27), while keeping the functions ξ_R and ξ_G independent and free. The set $\mathbb{U}$ hence describes all possible pairs of utility functions in the limit $v \to 0$.

4.2 Link function

For each pair of utility functions (u_R, u_G) of $\mathbb{U}$, there exists one corresponding link function $L_{[u_R,u_G]}$ such that $\Delta u = \Delta L$ for each possible move. With the notations introduced at the previous section, this function L can be written as:

$$L(x) = \sum_q \Big(\sum_{r=0}^{n_{qR}-1} \xi_R(r/H) + \sum_{g=0}^{n_{qG}-1} \xi_G(g/H) + \sum_{r=0}^{n_{qR}-1} \sum_{g=0}^{n_{qG}-1} \xi(r/H, g/H) \Big) \tag{28}$$

The proof is rather straightforward: it is sufficient to verify that the relation $\Delta u = \Delta L$ holds for any possible individual move.

In the thermodynamic limit, $L \simeq H \sum_q \ell(\rho_{qR}, \rho_{qG})$, with

$$\ell(\rho_R, \rho_G) = \int_0^{\rho_R} d\rho' \xi_R(\rho') + \int_0^{\rho_G} d\rho' \xi_G(\rho') + \int_0^{\rho_R} d\rho' \int_0^{\rho_G} d\rho'' \xi(\rho', \rho'') \tag{29}$$

Starting from this expression of the link function, it is straightforward to derive an expression for the potential $F(x)$. Once a specific pair of utility function (u_R, u_G) is chosen, the stationary configuration can be computed thanks to phase transition methods.

4.3 Number *vs* fraction of similar neighbors' dependent utility functions

Up to now, we have defined the utility as a function of the numbers (or densities) of similar and dissimilar agents. In the (Schelling) literature, one often finds the utility defined as a function of the fraction of similar neighbors. Denoting by $\rho_{qV} = 1 - \rho_{qR} - \rho_{qG}$ the density of vacant cells in block q, the fraction of similar neighbors for a red agent living in block q would be e.g.:

$$s_R = \frac{n_{qR}}{n_{qR} + n_{qG}} = \frac{\rho_R}{1 - \rho_V} \tag{30}$$

The question then arises of whether our formalism applies to such a definition. It can easily be checked that, in general, a pair (u_R, u_G) of utility functions defined on the fraction of similar neighbors do not satisfy the conditions (27). However, Eq. (30) shows that, for a block with a low density of vacant cells, $s_R = \rho_R$ at first order in ρ_V. Therefore, descriptions of the model based on s_R or ρ_R become similar if the global vacancy rate is low. In practice, even for vacancy rates as high as 10%, simulations run with utility functions depending on $s_R =$ instead of ρ_R lead to qualitatively equivalent stationary configurations, as shown by Fig. 10.

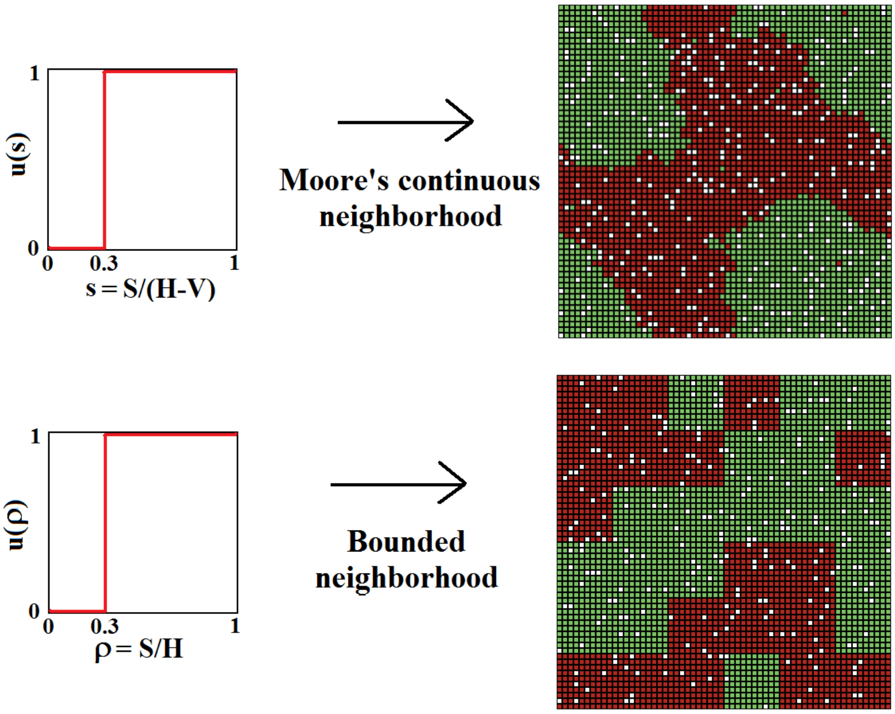

Figure 10: Snapshots of stationary configurations obtained by simulations. **Up**: the agent's utility function depends on their fraction of similar neighbors. The agent's neighborhood is composed of the H cells surrounding them. **Bottom**: the agent's utility functions depend on the number of similar neighbors. The neighborhood is composed of the H cells of the block they are living in. The values used in the figures are : $Q = 36$, $H = 100$, vacancy rate: 10%.